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Theory of Rotational Dynamic Measurement of Plastics

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Synopsis

A theory which postulates a condition of combined constraints of constant stress and constant strain is developed for the rotating cantilever beam measurement of dynamic mechanical properties of rigid plastics. The theory provides operating equations for the rotating beam instrument of Maxwell equipped with a biaxial strain gage dynamometer. The storage component of the dynamic Young's modulus $E' = 64\ L^3F_1/3\pi d^4(\Delta X_t - K_1F_1)$ and the mechanical loss tangent $\tan\gamma = [F_2(\Delta X_t - K_1F_1) + K_2F_1F_2]/[F_1(\Delta X_t - K_1F_1) - K_2(F_2)^2]$ are expressed in terms of the length L and diameter d of the circular rod specimen; the bending spring constants of the biaxial dynamometer K_1, K_2; the imposed dynamometer displacement (ΔX_t); and the cantilever beam storage and loss response forces F_1, F_2.

Introduction

The advancement of instrumental techniques for dynamic mechanical properties of plastics has made analysis of viscoelastic properties of rigid polymers under sinusoidally oscillating stress and strain a practical and important part of mechanical property characterization. Two versatile techniques for measurement of these dynamic properties of plastics have been described by Maxwell.[1,2] Both of these techniques involve the flexural deformation of a rotating cantilever beam. The rotating cantilever beam measurement is a direct stress-strain method which provides a broad and continuously variable frequency range under conditions of constant imposed stress[1] or strain.[2]

The chuck mounted cantilever beam specimen of circular cross section is illustrated in Figure 1*A* under a static flexural load F_1 and deflection ΔX_s. The constant dynamic stress condition is illustrated in Figure 1*B*, which presents an end view of the rotating specimen. In Figure 1*B* the applied constant force F_1 imposes the constant stress condition, and the respondant deflections ΔX_s and ΔY_s characterize the sample dynamic properties. The applied sample deflection (ΔX_s) characterizes the constant strain condition in Figure 1*C*, where the resultant forces F_1 and F_2 define the dynamic response of the specimen.

The theoretical examination discussed here recognizes the conditions represented in Figures 1*B* and 1*C* as special cases in a more general condition of dynamic stress and strain in a rotating cantilever beam specimen. Definition of this more general stress-strain condition is necessary properly

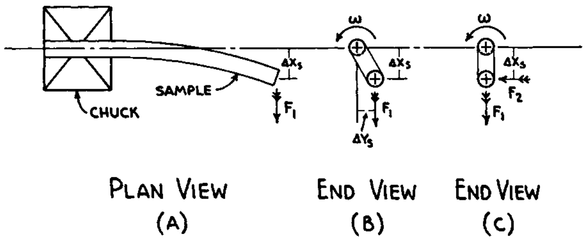

Fig. 1. Diagrams of (A) cantilever beam of circular cross section supported in a chuck at one end and statically deflected at the free end a distance ΔX_s by a force F_1; (B) rotating (ω) free end of the cantilever beam under a constant stress imposed by force F_1 and producing resultant in-phase deflection ΔX_s and out-of-phase deflection ΔY_s; (C) Rotating (ω) free end of the cantilever beam under a constant strain (ΔX_s) and producing resultant in-phase force P_1 and out-of-phase Force P_2.

to isolate the true specimen response in terms of dynamic properties from the gross measurement result. This general definition is also useful in instrument design where an objective may be either a constant stress or constant strain type measurement condition. In our laboratories the equations developed here form part of a digital computer program utilized in processing data from this measurement.

Definition of Combined Constraints

Each of these pure conditions of constant stress and constant strain may be considered as special cases of a more general situation of stress and strain within the rotating beam which can be termed a condition of combined constraints. These two special conditions and the more general condition of stress and strain are schematically depicted in Figure 2 in terms of the external forces and deflections and internal stress and strain profiles viewing the free end face of the rotating beam.

As Figure 2 indicates, the condition of constant stress causes a rotation of the neutral axis of internal strain through some lead angle δ_b in advance of the constrained neutral axis of internal stress imposed by the force F_1. Conversely the constant strain condition rotates the neutral axis of internal stress through some lag angle δ_a back from the constrained neutral axis of internal strain imposed by the displacement ΔX_s. The general condition of combined constraints can be viewed as the linear combination effect of imposing a condition of both constant stress and constant strain. The resultant effect upon internal stress and strain is a lead angle δ_b for the neutral axis of internal strain and a lag angle of δ_a for the neutral axis of stress away from a neutral axis defining the neutral axes of the pure single constraint conditions of both constant stress and constant strain. The resultant rotation of the neutral strain from neutral stress axis is then the angular sum, $\delta = \delta_a + \delta_b$. The defining relations for the neutral axis rotations in terms of the definition of the dynamic Young's modulus:

$$E^*(\omega) = E'(\omega) + iE''(\omega) \qquad i = (-1)^{1/2} \qquad (1)$$

where E^* is the complex dynamic modulus, E' the storage (or in-phase) component, and E'' the loss (or out-of-phase) component at some constant frequency ω, are presented also in Figure 2.

Maxwell has stated the advantages of the rotating beam measurement that approaches the condition of constant strain measurement. This condition is accomplished by applying the deflection to the rotating beam free end through a biaxial load sensing strain gage by dynamometer. Several instruments of this type are in operation in our laboratories.[3] Similar

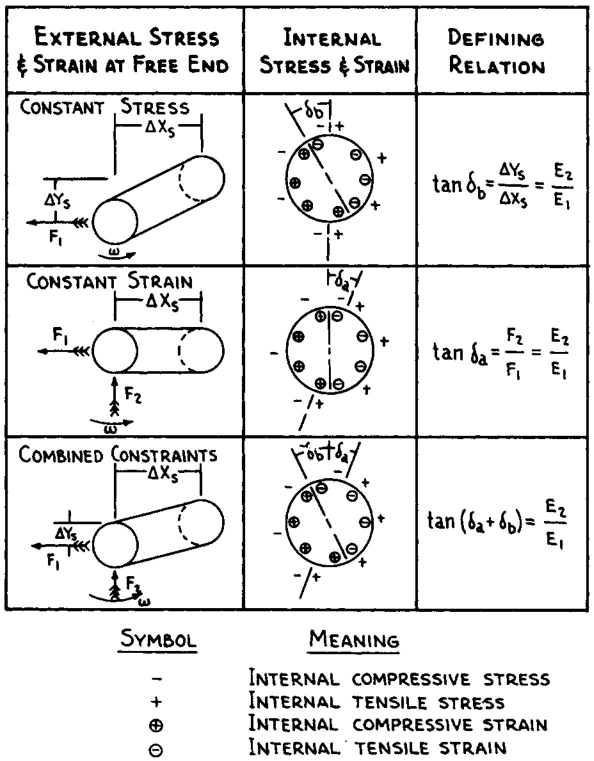

Fig. 2. Geometry of internal stress and strain.

apparatus is also commercially available.[4] This definition of combined constraints will be developed with respect to the operating equations of this type of instrument. Since the dynamometer acts in a mechanical series fashion with the sample, the combined interaction of dynamometer and sample must be considered in this analysis. In other words, we consider the fact that the measurement deflection ΔX_t is imposed upon the dynamometer and acts through the dynamometer upon the sample. The total deflection ΔX_t is then composed of both an instrument and a specimen deflection.

Theory

Several simplifying assumptions are made concerning the stresses and strains within the rotating cantilever beam: (*1*) the sample is strained in the region of linear stress–strain response; (*2*) bending is uniform through the length of the sample; (*3*) no torsional strain is present.

Following the geometry of the schematics of Figure 1 we establish by convention: (*a*) all in-phase (X axis) components of force and displacement are positive; (*b*) all out-of-phase (Y axis) components of force and displacement are negative for clockwise rotation and positive for counter clockwise rotation.

Table I presents the nomenclature applied in the development of the operational equations. From the geometry of the system:

TABLE I
Nomenclature

Symbol	Definition
ΔX_t	Total X-axis deflection of rotating beam and dynamometer
ΔX_s	X-axis deflection of beam free end
ΔX_c	X-axis deflection of dynamometer
ΔY_s	Y-axis deflection of sample free end
ΔY_c	Y-axis deflection of dynamometer
F_1	X-axis component of bending force
F_2	Y-axis component of bending force
K_1	X-axis spring constant of dynamometer
K_2	Y-axis spring constant of dynamometer
I	Moment of inertia of a rod of circular cross section about its neutral axis
d	Diameter of circular rod
L	Length of cantilever beam sample
$E^*(\omega)$	Complex Young's modulus
$\|E^*(\omega)\|$	Absolute Young's modulus
E'	Storage (in-phase) modulus
E''	Loss (out-of-phase) modulus
$\tan \delta$	Loss tangent
ω	Frequency

$$\Delta X_t = \Delta X_s + \Delta X_c \tag{2}$$

$$\Delta Y_s = \Delta Y_c \tag{3}$$

From the elastic character of the dynamometer:

$$\Delta X_c = K_1 F_1 \tag{4}$$

$$\Delta Y_c = K_2 F_2 \tag{5}$$

From the standard bending equation for a circular cross-sectioned cantilever beam:

$$\begin{aligned} \Delta X_s &= F_1 L^3/3E'I \\ &= 64 F_1 L^3/3\pi E' d^4 \end{aligned} \tag{6}$$

where

$$I = \pi d^4/64$$

Combining eqs. (2), (4), and (6) we obtain upon rearrangement:

$$E' = 64L^3F_1/3\pi d^4(\Delta X_t - K_1F_1) \quad (7)$$

which permits direct calculation of the storage modulus E' knowing the measurement variables F_1 and ΔX_t.

From the previously defined condition of combined constraints we may write eq. (8) for the loss tangent, tan δ:

$$\begin{aligned} \tan \delta &= \tan (\delta_a + \delta_b) \quad (8) \\ &= (\tan\delta_a + \tan\delta_b)/(1 - \tan\delta_a \tan\delta_b) \end{aligned}$$

where

$$\tan \delta_a = F_2/F_1 \quad (9)$$

$$\tan \delta_b = \Delta Y_s/\Delta X_s \quad (10)$$

Substituting eqs. (9) and (10) into eq. (8) we obtain:

$$\tan \delta = (F_2\Delta X_s + F_1\Delta Y_s)/(F_1\Delta X_s - F_2\Delta Y_s) \quad (11)$$

Substituting eq. (4) into eq. (2) we obtain upon rearrangement:

$$\Delta X_s = \Delta X_t - K_1F_1 \quad (12)$$

Substituting eq. (5) into eq. (3) we obtain:

$$\Delta Y_s = K_2F_2 \quad (13)$$

Upon substitution of eqs. (12) and (13) into eq. (11) we obtain the desired expression for the loss tangent:

$$\tan \delta = [F_2(\Delta X_t - K_1F_1) + K_2F_1F_2]/[F_1(\Delta X_t - K_1F_1) - K_2(F_2)^2] \quad (14)$$

Equation (14) then permits direct calculation of the loss tangent tan δ in terms of the experimental variables ΔX_t, F_1, and F_2.

Since the X-axis and Y-axis forces measured by the dynamometer are ordinarily recorded in terms of scale units it is convenient to introduce the following proportionality relations:

$$F_1 = \phi_1P_1 \quad (15)$$

$$F_2 = \phi_2P_2 \quad (16)$$

where P_1 and P_2 are the recorder deflection in scale units and ϕ_1 and ϕ_2 are the conversion constants relating them to their respective force values. Substituting eqs. (15) and (16) into eqs. (7) and (14), respectively we may write the operating equations of the rotating beam instrument as follows:

$$E'(\omega) = 64L^3\phi_1P_1/3\pi d^4(\Delta X_t - K_1\phi_1P_1) \quad (17)$$

$$\tan \delta(\omega) = [\phi_2 P_2(\Delta X_t - K_1\phi_1 P_1) + K_2\phi_1\phi_2 P_1 P_2]/[\phi_1 P_1(\Delta X_t - K_1\phi_1 P_1) - K_2(\phi_2 P_2)^2] \quad (18)$$

$$E''(\omega) = E' \tan \delta \quad (19)$$

$$|E^*(\omega)| = [(E')^2 + (E'')^2]^{1/2} \quad (20)$$

Equations (17) and (18) completely define the dynamic mechanical properties of the rotating cantilever beam at some constant frequency ω and temperature T. These calculated parameters E' and $\tan \delta$ may be utilized through eqs. (18) and (19) further to express these dynamic mechanical properties based on the rotating cantilever beam measurements in terms of standard nomenclature provided for linear viscoelastic materials.[5]

Discussion

From the preceding theory we may now examine the design criteria for constant stress and constant strain type dynamic measurement. Additionally we may examine the character of instrument response which must exist during the course of dynamic measurement through a glass-rubber transition region where dynamic moduli $|E^*|$ and E' change by magnitude factors of 100 to 10,000.

Combining eqs. (4) and (5) and differentiating we obtain:

$$d(\Delta X_c)/d(\Delta X_s) = K_1(3E'I/L^3) \quad (21)$$

and eqs. (6), (9), and (10) we may write:

$$\tan \delta_b/\tan \delta_a = K_2\,(3E'I/L^3) \quad (22)$$

We may define an approach to a condition of constant stress or constant strain measurement in terms of eqs. (21) and (22). These criteria and the instrument design constants which provide an approach, to within a 1% precision, to a pure constant stress or constant strain type measurement are indicated in Table II.

Several points of information may be drawn from Table II. Apparently for either constant stress or strain measurement the X and Y axis spring constants should be of equivalent magnitude, thus $K_1 \simeq K_2$. Table II also

TABLE II
Criteria for Constant Stress or Constant Strain Dynamic Measurement

Measurement condition	Prerequisite to condition fulfillment within 1% precision	Associated instrument design criteria
Constant stress		
Storage modulus	$d(\Delta X_c)/d(\Delta X_s) \geqslant 100.0$	$K_1 \geqslant 100\,(L^3/3\,E'I)$
Loss tangent	$\tan \delta_b/\tan \delta_a \geqslant 100.0$	$K_2 \geqslant 100\,(L^3/3\,E'I)$
Constant strain		
Storage modulus	$d(\Delta X_c)/d(\Delta X_s) \leqslant 0.01$	$K_1 \leqslant 0.01\,(L^3/3\,E'I)$
Loss tangent	$\tan \delta_b/\tan \delta_a \leqslant 0.01$	$K_2 \leqslant 0.01\,(L^3/3\,E'I)$

indicates the spring constants K_1, K_2 for a constant stress measurement must be 10,000 times higher than for a constant strain measurement for a given sample geometry and storage modulus. Thus the dynamometer design required for either type of measurement is highly specialized to the type of measurement selected. For constant stress measurement the dynamometer has a soft response compared to the sample. Conversely, constant strain measurement requires a high stiffness compared to the sample.

Table II also indicates how the measurement condition, with fixed values of K_1, K_2, varies with changes in the dynamic storage modulus E' that occur with temperature or frequency variation. An instrument designed for constant stress type measurement of glassy state response may well produce a constant strain type measurement as the storage modulus E' diminishes with increased temperature and decreased frequency.

A constant stress type instrument must then be designed, by the criteria of Table II, relative to lowest value of E' one anticipates encountering. A constant strain type instrument must preselect values of K_1 and K_2 with respect to the highest values of E' one may encounter. This manner of selecting K_1 and K_2 will preserve the intended character of measurement even though the storage modulus E' and other associated dynamic mechanical properties of the material change.

It is important to recognize here that design considerations of the instrument can only direct the mode of measurement toward a constant stress or strain type measurement. Either type of measurement in its pure form is difficult to accomplish. It is advantageous, therefore, to recognize the real condition of combined constraints and to apply eqs. (17–20) in the calculation of results.

References

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Résumé

On a développé une théorie qui postule un écart combiné à la tension constante et à la déformation constante pour la mesure des propriétés mécaniques dynamiques des plastiques rigides à l'aide d'appareil à potence tournante. Cette théorie propose des équations applicables à l'appareil de Maxwell équipé d'une jauge de contrante biaxiale. La partie réelle du module de Young dynamique $E' = 64\, L^3F_1/3\pi d^4(\Delta X_t - K_1F_1)$ et la tangente d'amortissement mécanique $\tan \gamma = [F_2(\Delta X_t - K_1F_1) + K_2F_1F_2]/[F_1(\Delta X_t - K_1F_1) - K_2(F_2)^2]$ sont exprimés en termes de longuer L et de diamètre d de la lame circulaire; des constantes élastiques de flexion du dynamomètre biaxial K_1, K_2; du déplacement imposé au dynamomètre (ΔX_t); et des forces de réponse F_1, F_2 à la dissipation et à la conservation.

Zusammenfassung

Eine Theorie, welche eine Bedingung kombinierter Anwendung konstanter Spannung und konstanter Verformung postuliert, wurde für die Messung der dynamisch- mechanischen Eigenschaften starrer Kunststoffe mit dem rotierenden, freitragenden Stab entwickelt. Diese Theorie liefert Arbeitsgleichungen für das Rotationsstabinstrument von Maxwell, das mit einem biaxial- verformungsmessenden Dynamometer ausgestattet wurde. Der Imaginärteil des dynamischen Youngmodul $E' = 64L^3F_1/3\pi d^4\,(\Delta X_t - K_1F_1)$ und der mechanische Verlustattangense $\tan\gamma = [F_2(\Delta X_t - K_1F_1) + K_2F_1F_2]/[F_1(\Delta X_t - K_1F_1) - K_2(F_2)^2]$ werden durch die Länge L und den Durchmesser d der kreisstabförmigen Probe, die Biegefederkonstanten des biaxialen Dynamometers K_1, K_2, die angewandte Dynamometerverschiebung (ΔX_t) und die Imaginär und Verlustreaktionskräfte F_1, F_2 des freitragenden Stabesausgedrückt.

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